

RENORMALIZATION FOR A CLASS OF DYNAMICAL SYSTEMS: SOME LOCAL AND GLOBAL PROPERTIES

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ABSTRACT. We study the period doubling renormalization operator for dynamics which present two coupled laminar regimes with two weakly expanding fixed points. We focus our analysis on the potential point of view, meaning we want to solve

$$V = \mathcal{R}(V) := V \circ f \circ h + V \circ h,$$

where f and h are naturally defined. Under certain hypothesis we show the existence of a explicit “attracting” fixed point V^* for \mathcal{R} . We call \mathcal{R} the renormalization operator which acts on potentials V . The log of the derivative of the main branch of the Manneville-Pomeau map appears as a special “attracting” fixed point for the local doubling period renormalization operator.

We also consider an analogous definition for the one-sided 2-full shift Σ (and also for the two-sided shift) and we obtain a similar result. Then, we consider global properties and we prove two rigidity results. Up to some weak assumptions, we get the uniqueness for the renormalization operator in the shift.

In the last section we show (via a certain continuous fraction expansion) a natural relation of the two settings: shift acting on the Bernoulli space $\{0, 1\}^{\mathbb{N}}$ and Manneville-Pomeau-like map acting on an interval.

1. INTRODUCTION

1.1. General presentation. The period doubling renormalization operator was introduced by M. Feigenbaum and by P. Coullet and C. Tresser (see [4] [7] [8] [21] [22]) for the study of a certain class of one-dimensional dynamical systems. We recall that for $f : [0, 1] \leftarrow$, the renormalization of f is defined by

$$(1) \quad \mathcal{R}(f)(x) = h^{-1} \circ f^2 \circ h(x),$$

where h is an affine map defined on $[0, 1]$.

This operator \mathcal{R} acts on dynamical transformations f .

A difficult problem is to find all the functions f which solve the equation $\mathcal{R}(f) = f$. In that direction, the renormalization conjecture is that in the proper class of maps, the period doubling renormalization operator has a unique fixed point which is hyperbolic, with a one-dimensional unstable manifold, and with a codimension one stable manifold consisting of systems at the transition to chaos (see [4] [5]).

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The goal of this article is to present some investigations in view to solve $\mathcal{R}(f) = f$ and to present some rigidity results.

Taking derivative in (1), and keeping in mind that h is affine, we get

$$f'(f \circ h(x))f' \circ h(x) = f'(x).$$

Then, taking the logarithm in this last equation and setting $V(x) := \log f'(x)$, we finally get

$$(2) \quad V(f(h(x))) + V(h(x)) = V(x).$$

Here we are interested in finding the solution V of the above equation.

This way of studying the renormalization operator is, in our view, more in the spirit of the setting of Statistical Mechanics, where the renormalization is looking for potentials and not looking for different dynamics (see *e.g.* [6]).

Our main motivation is chapter 5 of the book [19], where the renormalization is associated to the existence of a weakly expanding fixed point. We look for the problems raised in [19] but from the point of view of potentials, not from their point of view (which is in some sense purely dynamical). We give rigorous mathematical proofs. We point out that the purely dynamical problem is harder to deal.

Unfortunately the map f still appears in (2), which is a real obstacle to solve the equation. In this article, we are only interested by maps whose dynamics is conjugated to the full and one-sided shift σ acting on the Bernoulli space Σ with two symbols. Indeed, one deep problem to solve the renormalization conjecture is the huge number of different sorts of dynamical systems one has to deal with. However, a partial solution can be given when we assume some restrictions on the class of dynamics we are studying. Here, we are interested in Manneville-Pomeau like maps: each map has 2 coupled laminar regimes, with two fixed and weakly expanding points. These dynamics are canonically conjugated with (Σ, σ) .

In that case, the lifted equation of (2) in Σ is on the form

$$(3) \quad V(\sigma(H(x))) + V(H(x)) = V(x),$$

where $H : \Sigma \rightarrow \Sigma$.

In the case of the shift, we will consider a renormalization operator \mathcal{R} acting on potentials V :

given $V : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}$ in a certain natural class of potentials \mathcal{F}_γ , indexed by a real parameter γ , for any $x := \underbrace{(0, \dots, 0)}_{c_1} \underbrace{(1, \dots, 1)}_{c_2} \underbrace{(0, \dots, 0)}_{c_3} (1, \dots)$,

we set

$$\mathcal{R}(V)(x) = V(\underbrace{(0, \dots, 0)}_{2c_1} \underbrace{(1, \dots, 1)}_{c_2} \underbrace{(0, \dots, 0)}_{c_3} (1, \dots)) + V(\underbrace{(0, \dots, 0)}_{2c_1+1} \underbrace{(1, \dots, 1)}_{c_2} \underbrace{(0, \dots, 0)}_{c_3} (1, \dots)).$$

We show that the potential V^* defined in the following way: $V^*(x) = \gamma \log \frac{k+1}{k}$, for any x in the cylinder set $\underbrace{[000 \dots 00]}_k 1$, $k \in \mathbb{N}$, is a fixed point for \mathcal{R} . Moreover, for

any $V \in \mathcal{F}_\gamma$, we have that $\lim_{n \rightarrow \infty} \mathcal{R}^n(V) = V^*$

One can ask if other kind of renormalization operators could be considered (giving similar results).

Theorem A. *For the one-side shift, there is a unique renormalization operator (up to a constant parameter $a \in \mathbb{N}$).*

In this direction, we basically show that there exists a unique type of maps $H : \Sigma \leftrightarrow$, and a unique type of “good” potential V which satisfy (3). From here, the problem of solving (2) lies in the study of good projections from Σ onto $[0, 1]$.

We also analyze a family of Manneville-Pomeau maps and a local renormalization operator (defined in a different but similar way).

Each map of the one parameter family of Manneville-Pomeau like we study has 2 coupled laminar regimes, with two fixed and weakly expanding points. It is defined by

$$\begin{cases} f_t(x) = \frac{x}{(1-x^t)^{1/t}} = (\frac{x^t}{1-x^t})^{1/t}, & \text{if } 0 \leq x \leq \frac{1}{2^{1/t}}, \\ f_t(x) = (2 - \frac{1}{x^t})^{1/t}, & \text{if } \frac{1}{2^{1/t}} < x \leq 1, \end{cases}$$

where $t > 0$ is a real parameter.

Note that, for these maps, the renormalization makes sense only for the basins of backward-attraction of the two weakly expanding fixed points.

One important result in the setting of Manneville-Pomeau transformation is:

Theorem B. *For each t , f_t is an hyperbolic fixed point for the doubling period renormalization operator (restricted to each basin). The stable leaf \mathcal{F}_t is given by dynamics with the same germs than f_t in 0 (and in 1)*

For each t there is a unique f_t -invariant measure absolutely continuous with respect to Lebesgue measure. In classical (non)-uniformly hyperbolic dynamics, such a measure is referred as a *physical measure*. It is so, because one can actually “see” the convergence in Birkhoff averages for points. Now, it is well-known that the nature (finite/infinite) of this invariant measure is related and only depends on the nature of the germ of the dynamics close to the fixed and weakly hyperbolic point (see [15]).

Finally, in the last section (which covers global aspects), we show (via a certain continuous fraction expansion) a natural relation of the two settings: shift acting on the Bernoulli space $\{0, 1\}^{\mathbb{N}}$ and Manneville-Pomeau-like map acting on an interval.

1.2. Structure of the paper and results. This paper is organized in the following way.

In Section 2 we study the local renormalization. In Subsection 2.1 we show the fixed point property for the renormalization operator associated to Manneville-Pomeau transformations and also Theorem B. In Subsection 2.2 we consider the one-sided shift and we define there the natural renormalization operator with respect to the class of dynamics we are considering. As a by-product we extend the operator to the 2-side case in Subsection 2.3, and then consider a kind of two dimensional bijective Baker Manneville-Pomeau map in Subsection 2.4.

In general terms, this section (which consider local properties) can be considered as associated to the dynamics restricted to the basin of attraction (backward) of a weakly expanding fixed point. In the shift, each different laminar regime (where the renormalization operator acts) is associated to a different parameter γ . For the Manneville-Pomeau map the parameter t plays the role of the γ . In Phase Transition Theory the correspondence of the two settings (shift versus MP) is given by $\gamma = 1 + \frac{1}{t}$ (in [9] [14], the Manneville-Pomeau map is defined in a slightly different way and indexed by a parameter s , the correspondence of the parameters in the two cases, here and there, is given by $t = s - 1$).

In the second part of the paper, we investigate global properties, and get two results of rigidity.

First, we prove Theorem A in Section 3, that is, there exists a unique renormalization operator (up to an integer positive parameter a) for the shift which respects the class of dynamics we are considering (two coupled laminar regimes with two fixed and weakly repelling points).

Several other results in the paper are for a general positive parameter $a \in \mathbb{R}$ which appears in the late sections.

Diversity of the dynamics can therefore only follow from the choices of the laminar parameter γ and from the choices of the conjugacies with the interval.

In Section 4 we consider a family of conjugacies between the shift Σ and the interval. Instead of the laminar parameter γ as above, we consider a parameter denoted by α . An associated value β appears; it corresponds to a different coupled laminar regime. For technical reasons, an extra positive parameter $a \in \mathbb{R}$ also appears. We then introduce a new continuous fraction expansion and a pair of Gauss maps associated to α and β . In the last subsection we study this family of transformation and get the second rigidity result: up to some scaling renormalization, all these maps are Manneville-Pomeau maps. In other words, the renormalization operator on the shift (which is uniquely defined in some sense) is naturally associated, via a change of coordinates $\theta : \Sigma \rightarrow [0, 1]$ (associated to a continuous fraction expansion), to a Manneville-Pomeau-like map (depending on certain parameters).

We point out that our setting has a different nature from the usual one consider for the dynamics of one-dimensional transformations as in [4] [5] [7] [8] [10] [16] [21] [22].

The renormalization procedure we will consider here is associated to the occurrence of dynamical phase-transitions (see [9], [14], [17], [15], [18], [19], [23], [11] [13]). Our proof do not require any of the results on these papers. Some other mathematical references on phase transitions are [6] [11] [12] [20].

2. THE LOCAL RENORMALIZATION OPERATOR

2.1. The Manneville-Pomeau model. Consider

$$\begin{cases} f(x) = \frac{x}{1-x}, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ f(x) = 2 - \frac{1}{x}, & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$

Note that one branch above is obtained from the other by the change of coordinate $x \rightarrow (1 - x)$.

Consider also for $t \geq 0$,

$$\begin{cases} f_t(x) = \frac{x}{(1 - x^t)^{1/t}} = (\frac{x^t}{1 - x^t})^{1/t}, & \text{if } 0 \leq x \leq \frac{1}{2^{1/t}}, \\ f_t(x) = (2 - \frac{1}{x^t})^{1/t}, & \text{if } \frac{1}{2^{1/t}} < x \leq 1, \end{cases}$$

For instance, in the first injective domain of

$$f_t(x) = \frac{x}{(1 - x^t)^{1/t}},$$

if $h_t(x) = x^t$, and, if we denote $f_1 = f$, then we have $f_t = h_t^{-1} \circ f \circ h_t$. Same thing for the other injective domain.

From this property one can get invariant a.c.i.m for each f_t (just change coordinates).

In this way we have a natural partition by fundamental domains for the branch of f_t in $(0, (1/n)^{1/t})$ by $(\frac{1}{3^{1/t}}, \frac{1}{2^{1/t}}), \dots, (\frac{1}{k^{1/t}}, \frac{1}{(k+1)^{1/t}}), \dots$

For a given y , the two inverse branches by f_t are $x_{1,t}(y) = \frac{y}{(1 + y^t)^{1/t}}$ and $x_{2,t}(y) = (\frac{1}{2 - y^t})^{1/t}$.

The image of $x_{1,t}$ is $[0, (0.5)^{1/t}]$ and the image of $x_{2,t}$ is $[(0.5)^{1/t}, 1]$.

Note that $f'_1(x) = \frac{1}{(1-x)^2}$ for $x \in (0, 0.5)$ and $f'_1(x) = \frac{1}{x^2}$ for $x \in (0.5, 1)$

Moreover, by the chain rule, $f'_t(x) = \frac{1}{(1 - x^t)^{1+1/t}}$ for $x \in (0, 0.5^{1/t})$ and $f'_t(x) = \frac{(2x^t - 1)^{-1+1/t}}{x^2}$ for $x \in (0.5^{1/t}, 1)$

We point out the main property of f_t :

$$(4) \quad f_t^2(\frac{x}{2^{1/t}}) = (f_t \circ f_t)(\frac{x}{2^{1/t}}) = \frac{1}{2^{1/t}} f_t(x).$$

One can see by induction that

$$(5) \quad f_t^j(x) = \frac{1}{(\frac{1}{x^t} - j)^{1/t}}.$$

Definition 2.1. For a given value $t \geq 0$ we denote \mathcal{F}_t the set of non-negative continuous functions $V : [0, 1] \rightarrow \mathbb{R}$ such that $V(x) \sim (1 + \frac{1}{t})x^t$ when $x \sim 0$.

Above $V(x) \sim (1 + \frac{1}{t})x^t$ means $V(x) = (1 + \frac{1}{t})x^t + O(x^{t+\varepsilon})$.

Definition 2.2. The renormalization operator \mathcal{R} acts on the set of functions V on \mathcal{F}_t by means of

$$\mathcal{R}(V)(x) = V(f_t(\frac{x}{2^{1/t}})) + V(\frac{x}{2^{1/t}}).$$

We point out that our model is not just a "log" version of the one described by Schuster and Just [19]. This is so because we are using here the dynamics of f_t , given a priori. Anyway, our result is in the spirit of the setting of Statistical

Mechanics where the renormalization is for potentials and not for different dynamics [6]. In other words, we look for fixed point potentials and not for fixed point transformations.

By recurrence and using (4) one can easily see that

$$\mathcal{R}^n(V)(x) = [S_{2^n}(V)]\left(\frac{x}{2^{n/t}}\right) = \sum_{j=0}^{2^n} V(f_t^j(\frac{x}{2^{n/t}})).$$

Note that from (5)

$$(6) \quad \mathcal{R}^n(V)(x) = \sum_{j=0}^{2^n} V\left(\frac{1}{(\frac{2^n}{x^t} - j)^{1/t}}\right).$$

Taking derivative of both sides of (4) one can see that

$$V^*(x) = \log f_t'(x) = -(1 + \frac{1}{t}) \log(1 - x^t),$$

is a fixed point for \mathcal{R} .

Our main interest is on universality type properties for the renormalization operator.

Theorem 2.1. *For any $V \in \mathcal{F}_t$, we have that*

$$\lim_{n \rightarrow \infty} \mathcal{R}^n(V) = V^*.$$

Proof. Let x be in $\left[\frac{1}{(m+1)^{1/t}}, \frac{1}{m^{1/t}}\right]$, with $m \geq 2$.

Then, $\frac{x}{2^{n/t}}$ belongs to $\left[\frac{1}{2^{n/t}(m+1)^{1/t}}, \frac{1}{2^{n/t}m^{1/t}}\right]$.

Hence, the smallest value for $(\frac{2^n}{x^t} - j)$, $j = 0, 1, \dots, 2^n$ is obtained when $j = 2^n$, and is larger than $2^n(2 - 1)$. Therefore each term $\frac{1}{\frac{2^n}{x^t} - j}$ is very close to 0, and it makes sense to approximate $V(f_t^j(\frac{x}{2^{n/t}}))$. Hence we have

$$(7) \quad \mathcal{R}^n(V)(x) = \sum_{j=0}^{2^n} V\left(\frac{1}{(\frac{2^n}{x^t} - j)^{1/t}}\right)$$

$$= (1 + \frac{1}{t}) \sum_{j=0}^{2^n} \frac{1}{(\frac{2^n}{x^t} - j)} + O\left(\frac{1}{(\frac{2^n}{x^t} - j)^{1+\varepsilon/t}}\right)$$

$$(8) \quad = (1 + \frac{1}{t}) \frac{1}{2^n} \sum_{j=0}^{2^n} \frac{1}{(\frac{1}{x^t} - \frac{j}{2^n})} + \frac{1}{2^{n\varepsilon/t}} O\left(\frac{1}{2^n} \sum_{j=0}^{2^n} \frac{1}{(\frac{1}{x^t} - \frac{j}{2^n})^{1+\varepsilon/t}}\right).$$

For a fixed x , the last expression is asymptotic to

$$(1 + \frac{1}{t}) \int_0^1 \frac{1}{(\frac{1}{x^t} - r)} dr = -(1 + \frac{1}{t}) [\log(\frac{1}{x^t} - r)]_0^1 = -(1 + \frac{1}{t}) \log(\frac{1}{1 - x^t}) = V^*(x).$$

□

We now explain how this is related to and proves Theorem B.

First, note that if $V(x) = c.x^{t'} + O(x^{t'+\varepsilon})$ with $t' < t$, then, assuming $c > 0$, the same proof than above yields that

$$\mathcal{R}^n(V) \rightarrow +\infty.$$

Therefore, only potentials V in \mathcal{F}_t can converge to the fixed point V^* .

Let us now assume that V belongs to \mathcal{F}_t . Let us set $g : [0, 1] \leftrightarrow$ such that $V = \log g'$. Then $\mathcal{R}^n V \rightarrow V^*$ is the expression of g belongs to the stable leaf of f_t for the action of \mathcal{R} . And as we said, this exactly means that g has the same germ than f_t .

2.2. The one-side shift Σ . We consider here the Bernoulli space $\Sigma = \{0, 1\}^{\mathbb{N}}$ and the shift acting on Σ .

We denote by $M_n \subset \Sigma$, for $n \geq 1$, the cylinder set $\underbrace{[000 \dots 00]_n}_n$ and by M_0 the cylinder set $[1]$. The ordered collection $(M_n)_{n=0}^\infty$ is a partition of Σ .

Definition 2.3. Consider \mathcal{F} the set of non-negative continuous functions $V : \Sigma \rightarrow \mathbb{R}$ which are constant in the set M_n , for all $n \geq 1$. We denote by a_n the value of V on each M_n . We further assume that $a_n = \frac{1}{n} + O(\frac{1}{n^{1+\varepsilon}})$, for some positive ε .

Definition 2.4. We define the renormalization operator in the following way:

For $x := (\underbrace{0, \dots, 0}_{c_1}, \underbrace{1, \dots, 1}_{c_2}, \underbrace{0, \dots, 0}_{c_3}, 1, \dots)$ we set

$$\begin{aligned} \mathcal{R}(V)(x) &= V((\underbrace{0, \dots, 0}_{2c_1}, \underbrace{1, \dots, 1}_{c_2}, \underbrace{0, \dots, 0}_{c_3}, 1, \dots)) + \\ &\quad V((\underbrace{0, \dots, 0}_{2c_1+1}, \underbrace{1, \dots, 1}_{c_2}, \underbrace{0, \dots, 0}_{c_3}, 1, \dots)). \end{aligned}$$

Note that the potential V^* , with value $\log \frac{k+1}{k}$ in M_k , is invariant by \mathcal{R} . Indeed we have

$$\log \frac{k+1}{k} = \log \frac{2k+1}{2k} + \log \frac{2k+1+1}{2k+1}.$$

Gibbs states (and its decay of correlation) of the potential V^* , with value $\gamma \log \frac{k+1}{k}$ in M_k , were analyzed in [9][14].

Theorem 2.2. Each $V \in \mathcal{F}$ is attracted by the renormalization operator \mathcal{R} to the fixed point V^* .

Proof. An easy computation, by induction, gives the formula

$$(9) \quad \mathcal{R}^n(V)(x) = S_{2^n}(V)(x_n)$$

where $x_n = (\underbrace{0, \dots, 0}_{2^n c_1 + 2^n - 1}, \underbrace{1, \dots, 1}_{c_2}, \underbrace{0, \dots, 0}_{c_3}, 1, \dots)$ and $S_k(V)$ is the Birkhoff sum $V(\cdot) + V \circ \sigma(\cdot) + \dots + V \circ \sigma^{k-1}(\cdot)$.

Equation (9) yields for $x \in M_{c_1}$

$$\begin{aligned} \mathcal{R}(V)(x) &= \sum_{j=0}^{2^n-1} a_{2^n c_1 + j} = \sum_{j=0}^{2^n-1} \frac{1}{(2^n c_1 + j)} + O\left(\frac{1}{(2^n c_1 + j)^{1+\varepsilon}}\right) \\ &= \frac{1}{2^n} \sum_{j=0}^{2^n-1} \frac{1}{(c_1 + \frac{j}{2^n})} + \frac{1}{2^{n\varepsilon}} O\left(\frac{1}{2^n} \sum_{j=0}^{2^n-1} \frac{1}{(c_1 + \frac{j}{2^n})^{1+\varepsilon}}\right) \end{aligned}$$

The first term in the right hand side is a Riemann sum, and converges, as $n \rightarrow \infty$, to $\int_0^1 \frac{1}{(c_1 + r)} dr$. Again the second term goes to zero.

Note that the integral $\int_0^1 \frac{1}{(c_1 + r)} dr$ is the same as $-\log \frac{c_1+1}{c_1}$. Thus, and in the same way as before, if the potential V satisfies the condition

$$a_k = \frac{1}{k} + \frac{1}{k^{1+\varepsilon}},$$

we have convergence of $\mathcal{R}^n(V)(x)$ to $V^*(x)$ when n goes to $+\infty$. □

Remark 1. A similar result can be obtained for $V^*(x) = \gamma \log \frac{k+1}{k} = a_k$, when $x \in M_k$, and $\gamma > 1$ is fixed. In this case we have to consider \mathcal{R} acting on the set \mathcal{F}_γ , as the set of V such that $a_k = \frac{\gamma}{k} + O(\frac{\gamma}{k^{(1+\varepsilon)}})$.

2.3. The two-sided shift $\hat{\Sigma}$. We denote $\hat{\Sigma} = \{0, 1\}^{\mathbb{Z}}$ and also denote each point in this set by $\langle y | x \rangle = \langle \dots y_2, y_1 | x_0, x_1, x_2 \dots \rangle$ where x is future and y is past. The shift $\hat{\sigma}$ is defined by

$$\hat{\sigma}(\langle \dots y_2, y_1 | x_0, x_1, x_2 \dots \rangle) = \langle \dots y_2, y_1, x_0 | x_1, x_2 \dots \rangle.$$

Definition 2.5. Consider \mathcal{F} the set of non-negative continuous functions $V : \Sigma \rightarrow \mathbb{R}$, which are constant in the sets of the form

$$M_m | M_n = \{ \langle y, x \rangle, x \in M_n, y \in M_m \},$$

for each pair $m, n \geq 1$. We denote by $a_{m,n} = V(m, n)$ the value of V on each $M_m \times M_n$. We further assume that $a_{m,n} = \frac{m+n}{(m-1)n} + O\left(\frac{1}{m^{1+\varepsilon_1}}\right) + O\left(\frac{1}{n^{1+\varepsilon_2}}\right)$, for positive ε_i .

Definition 2.6. We define the renormalization operator in the following way:
For

$$z := (\underbrace{0, \dots, 0}_{d_3}, \underbrace{1, \dots, 1}_{d_2}, \underbrace{0, \dots, 0}_{\zeta}) | (\underbrace{0, \dots, 0}_{c_1}, \underbrace{1, \dots, 1}_{c_2}, \underbrace{0, \dots, 0}_{c_3}, 1, \dots) \in M_\zeta \times M_{c_1},$$

we set

$$\mathcal{R}(V)(z) = V(\underbrace{0, \dots, 0}_{d_3}, \underbrace{1, \dots, 1}_{d_2}, \underbrace{0, \dots, 0}_{2\zeta-1}) | (\underbrace{0, \dots, 0}_{2c_1+1}, \underbrace{1, \dots, 1}_{c_2}, \underbrace{0, \dots, 0}_{c_3}, 1, \dots) +$$

$$V(\underbrace{0, \dots, 0}_{d_3}, \underbrace{1, \dots, 1}_{d_2}, \underbrace{0, \dots, 0}_{2\zeta} \mid \underbrace{0, \dots, 0}_{2c_1}, \underbrace{1, \dots, 1}_{c_2}, \underbrace{0, \dots, 0}_{c_3}, 1, \dots).$$

In order to simplify the notation we write

$$\mathcal{R}(V)(z) = V(2\zeta - 1, 2c_1 + 1) + V(2\zeta, 2c_1).$$

One can show that for $V \in \mathcal{F}$, and $z \in M_m | M_n$, we have that

$$\mathcal{R}^n(V)(z) = \sum_{k=0}^{2^n-1} V(2^n\zeta - 2^n + 1 + k, 2^n c_1 + 2^n - 1 - k).$$

It is easy to see that the potential given by: for each $z \in M_j | M_k$

$$V^*(z) = \log \frac{j(k+1)}{(j-1)k},$$

defines a fixed point potential for \mathcal{R} .

Theorem 2.3. *Each $V \in \mathcal{F}$ is attracted by the renormalization operator \mathcal{R} to the fixed point V^* .*

Proof. Given $V \in \mathcal{F}$, we have

$$\begin{aligned} \mathcal{R}^n(V)(z) &= \sum_{k=0}^{2^n-1} V(2^n\zeta - 2^n + 1 + k, 2^n c_1 + 2^n - 1 - k) \\ &= \sum_{k=0}^{2^n-1} \left(\frac{2^n(c_1 + \zeta)}{(2^n\zeta - 2^n + k)(2^n c_1 + 2^n - 1 - k)} + O\left(\frac{1}{(2^n\zeta - 2^n + k + 1)^{1+\varepsilon_1}}\right) \right. \\ &\quad \left. + O\left(\frac{1}{(2^n c_1 + 2^n - 1 - k)^{1+\varepsilon_2}}\right) \right) \\ &= \frac{1}{2^n} \sum_{k=0}^{2^n-1} \frac{(c_1 + \zeta)}{((\zeta - 1) + \frac{k}{2^n})((c_1 + 1) - \frac{k+1}{2^n})} + O(\frac{1}{2^{n\varepsilon_1}}) + O(\frac{1}{2^{n\varepsilon_2}}) \end{aligned}$$

Taking n large we get

$$\begin{aligned} &\frac{1}{2^n} \sum_{k=0}^{2^n-1} \frac{(c_1 + \zeta)}{((\zeta - 1) + \frac{k}{2^n})((c_1 + 1) - \frac{k+1}{2^n})} \sim \\ &\int_0^1 \frac{(c_1 + \zeta)}{((\zeta - 1) + x)((c_1 + 1) - x)} dx = \\ &(c_1 + \zeta) \int_0^1 \frac{1}{(\zeta - 1 + x)} + \frac{1}{(c_1 + 1 - x)} dx = \left[\log \frac{\zeta - 1 + x}{c_1 + 1 - x} \right]_0^1 = \\ &\log \left(\frac{\zeta(c_1 + 1)}{(\zeta - 1)c_1} \right) = V^*(z). \end{aligned}$$

□

2.4. The Baker Manneville-Pomeau bijective transformation. Using the notation of the first section, for a fixed value of t , consider

$$F_t : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1],$$

a bijective transformation such that satisfies for each x and y

$$F_t(x, f_t(y)) = (f_t(x), y).$$

Definition 2.7. For a given value $t \geq 0$, we denote \mathcal{F}_t the set of non-negative continuous functions $V : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ such that $V(x, y) = (1 + \frac{1}{t}) \log(\frac{1+x}{1-y}) + O(x^{1+\varepsilon_2}) + O(y^{1+\varepsilon_1})$ when $(x, y) \sim (0, 0)$.

In order to simplify the notation we consider here only the case $t = 1$. Similar results will be true for the general case $t \geq 0$. We use the notation $\mathcal{F}_1 = \mathcal{F}$.

Definition 2.8. The renormalization operator \mathcal{R} acts on the set of functions V on \mathcal{F} by means of

$$\mathcal{R}(V)(x, y) = V\left(\frac{2}{2+x}, \frac{y}{2-y}\right) + V\left(\frac{x}{2}, \frac{y}{2}\right)$$

In the same way as before one can show that

$$V^*(x, y) = \log\left(\frac{1+x}{1-y}\right),$$

is a fixed point for \mathcal{R} .

We leave for the reader the proof of the theorem:

Theorem 2.4. Each $V \in \mathcal{F}$ is attracted by the renormalization operator \mathcal{R} to the fixed point V^* .

3. UNIQUENESS OF THE RENORMALIZATION ON THE SHIFT

In this section, we consider global properties and we prove that, up to relatively weak assumptions, there exists a unique type of renormalization operator in the shift.

We first state a simple lemma:

Lemma 3.1. Let (X_1, T_1) and (X_2, T_2) be two conjugated dynamical systems. Let $\theta : X_1 \rightarrow X_2$ be the conjugacy. If H_1 satisfies $H_1^{-1} \circ T_1^2 \circ H_1 = T_1$, then $H_2 := \theta \circ H_1 \circ \theta^{-1}$ satisfies

$$H_2^{-1} \circ T_2^2 \circ H_2 = T_2.$$

Moreover if V_1 satisfies $V_1(T_1(H_1(x))) + V_1(H_1(x)) = V_1(x)$, then $V_2 := V_1 \circ \theta^{-1}$ satisfies

$$V_2(T_2(H_2(x))) + V_2(H_2(x)) = V_2(x).$$

In view of those results, it's meaningful to study maps H on the shift which satisfy the property $H^{-1} \circ \sigma^2 \circ H = \sigma$. More assumptions are necessary, if one wants to respect some other properties of the map $x \mapsto x/2$ in the interval. If 0^∞ in the shift represents the 0 of the interval, then $H(0^\infty) = 0^\infty$ needs to hold. Moreover the map H has to “increase”, which can be translated into “ H respect the lexicographic order in Σ ”. The uniqueness of the type of such map H follows from the next proposition:

Proposition 3.2. *Let H be an increasing function on the shift Σ_2 (for the lexicographic order), such that*

1. *for every $\underline{x} = (1, x_2, x_3, \dots)$, $H(\underline{x}) = (\underbrace{0, \dots, 0}_{a \text{ terms}}, 1, x_2, x_3, \dots)$, where $a \geq 1$;*
2. $H^{-1} \circ \sigma^2 \circ H = \sigma$,
3. $H(\underline{0}^\infty) = \underline{0}^\infty$

Then, for every $\underline{x} = (\underbrace{0, \dots, 0}_{n_0 \text{ terms}}, 1, x_{n_0+2}, \dots)$, we have $H(x) = (\underbrace{0, \dots, 0}_{2n_0+a \text{ terms}}, 1, x_{n_0+2}, \dots)$.

Proof. Note that the formula is correct for every \underline{x} on the form $(1, \dots)$. We first consider the case where $a \geq 2$. Note that we took $a = 1$ in a previous section where we considered the shift.

Let us pick some x , which necessarily has to be of the form $\underline{x} = (\underbrace{0, \dots, 0}_{n_0 \text{ terms}}, 1, x_{n_0+2}, \dots)$.

We assume $n_0 > 1$. We point out that $\sigma(\underline{x}) \geq \underline{x}$, because a “1” appears sooner in $\sigma(\underline{x})$ than in \underline{x} . Therefore we must have

$$(10) \quad H(\sigma(\underline{x})) > H(\underline{x}), \text{ if } x \neq \underline{0}^\infty, \underline{1}^\infty.$$

Now, $\sigma^{n_0}(\underline{x})$ belongs to the cylinder $[1]$, hence $H(\sigma^{n_0}(\underline{x})) = [\underline{ax}]$, where \underline{a} is the finite word $\underbrace{0, \dots, 0}_{a \text{ terms}}$, and $[\]$ is the concatenation of words in the shift. As we said before,

in the moment we are considering such $a \geq 2$. The constraint $H^{-1} \circ \sigma^2 \circ H = \sigma$, yields $\sigma^{2n_0} \circ H = H \circ \sigma^{n_0}$. Therefore

$$(11) \quad H(\underline{x}) = (\underbrace{?, \dots, ?}_{2n_0 \text{ terms}}, \underbrace{0, \dots, 0}_{a \text{ terms}}, 1, x_{n_0+2}, \dots),$$

where the first $2n_0$ digits are unknown.

As H has the increasing property, its image is in the cylinder $[0]$, and the first digit in (11) is 0. The property $H^{-1} \circ \sigma^2 \circ H = \sigma$, also means $\sigma^2 \circ H = H \circ \sigma$. Therefore, each odd unknown digit in (11) is 0.

Now, we prove that no even unknown digit can be 1. Let us assume that the second digit is 1. Doing the same work for $\sigma(\underline{x})$ (here we use $n_0 > 1$), we have

$$(12) \quad H \circ \sigma(\underline{x}) = (\underbrace{0, ?, \dots, 0, ?}_{2n_0-2 \text{ terms}}, \underbrace{0, \dots, 0}_{a \text{ terms}}, 1, x_{n_0+2}, \dots),$$

where each unknown digit at position $2p$ is the same digit than the digit in position $2p + 2$ in (11). To get these equalities, we again used $\sigma^2 \circ H = H \circ \sigma$.

If the second digit in (11) is a “1”, then to respect (10), the second digit in (12) must be a “1” too. Therefore, the cascade rule yields that each even unknown digit must be 1, in (11) and in (12). In that case, and as we assume $a \geq 2$, there will be a “1” in $H(\underline{x})$ in position $2n_0$, and a “0” for $H \circ \sigma(\underline{x})$, and the two words coincide before that position. Hence, $H(\sigma(\underline{x})) < H(\underline{x})$, which is impossible by (10). This proves that the assumption is false, and the second unknown digit in (11) must be a “0”.

Note that this also holds if $n_0 = 1$. Indeed, in that case we completely know $H \circ \sigma(\underline{x})$, by assumption (1) in the proposition. Therefore the above discussion means that for every $\underline{\xi} = (0, \dots)$, $H(\underline{\xi})$ starts with 3 “0”. Here again, the cascade rule between (11) and (12) yields that every even unknown digit is “0”.

To complete the proof of the proposition, we have to deal with the case $a = 1$. In that case, the assumption “the second unknown digit in (11) is 1” yields to

$$\begin{aligned} H(\underline{x}) &= \underbrace{(0, 1, \dots, 0, 1, 0, 1, \overset{\downarrow a}{0}, 1, x_{n_0+2}, \dots)}_{2n_0 \text{ terms}}, \\ H \circ \sigma(\underline{x}) &= \underbrace{(0, 1, \dots, 0, 1, \overset{\downarrow a}{0}, 1, x_{n_0+2}, \dots)}_{2n_0-2 \text{ terms}}. \end{aligned}$$

Hence, the unique possibility to respect the increasing property for H would be to alternate “0” and “1” for the tail of \underline{x} . But even in that case, this will be in contradiction with (10). This finishes the proof. \square

The conclusion is that each renormalization operator has to be of the form: take a fixed $a \in \mathbb{N}$, then given $V : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}$, for any $x := \underbrace{(0, \dots, 0, 1, \dots, 1)}_{c_1} \underbrace{0, \dots, 0}_{c_2} \underbrace{1, \dots, 1}_{c_3} \dots$,

we set

$$\mathcal{R}(V)(x) = V(\underbrace{(0, \dots, 0, 1, \dots, 1)}_{2c_1} \underbrace{0, \dots, 0}_{c_2} \underbrace{1, \dots, 1}_{c_3} \dots) + V(\underbrace{(0, \dots, 0, 1, \dots, 1)}_{2c_1+a} \underbrace{0, \dots, 0}_{c_2} \underbrace{1, \dots, 1}_{c_3} \dots).$$

4. DYNAMICS ON THE INTERVAL

4.1. Choices of the laminar parameter. From expression (9) in [2]

$$x = \frac{1}{(c_1 + 1) + \frac{1}{c_2 + \frac{1}{c_3 + \dots}}} = \theta(\bar{x}),$$

one gets a change of coordinates θ from the shift (where a point \bar{x} in Σ is denoted by $\bar{x} = \underbrace{(0, \dots, 0)}_{c_1} \underbrace{1, \dots, 1}_{c_2} \underbrace{0, \dots, 0}_{c_3} \dots \in \{0, 1\}^{\mathbb{N}}$) to the interval $[0, 1]$. This map preserves the lexicographic order.

Since we consider in the first place the potential, and not the dynamics, we point out that each c_i , for $i \geq 1$, and $c_1 + 1$ can be replaced by

$$\frac{1}{(1 + \frac{1}{n})^\alpha - 1},$$

where $n = c_1 + 1, c_2, c_3, \dots$ and $\alpha = 1$.

Now, remembering Remark 1, we note that each term $(1 + \frac{1}{c_i})^\alpha$ is some $e^{V^*(z_i)}$, where z_i is in relation with the orbit of the initial point under the actions of the two coupled and competitive laminar regimes.

The goal in this Section 4 is to obtain a more general change of coordinates of this sort, in order to take care of the choice of different possibilities of the parameter $\alpha \geq 1$.

4.2. Convergence of a new continued fraction expansion. In this section we define a new type of continued fraction.

Let $\alpha > 0$ be a real number. We define $g : (0, \infty) \rightarrow \mathbb{R}$, given by

$$g_\alpha(z) = g(z) = \frac{1}{(1 + \frac{1}{z})^\alpha - 1}.$$

For a fixed α , and when the meaning is clear, we omit the subscribe α in g_α , in order to make the formulas simpler.

We have for every $z \in (0, +\infty)$, $g'(z) = \frac{\alpha}{z^2} \frac{1}{((1 + \frac{1}{z})^\alpha - 1)^2} (1 + \frac{1}{z})^{\alpha-1}$, hence g is increasing. Moreover, $\lim_{z \rightarrow 0} g(z) = 0$ and $\lim_{z \rightarrow +\infty} g(z) = +\infty$. Therefore, for any given $y \in (0, \infty)$, there exists a $n_y \in \mathbb{N}$, such that

$$(13) \quad g(n_y) \leq y < g(n_y + 1).$$

Moreover, $g(z) = z^\alpha + o(z^\alpha)$ when z is close to 0, and, $g(z) = \frac{z}{\alpha} - \frac{\alpha - 1}{2} + O(\frac{1}{z})$, when z is close to $+\infty$.

Lemma 4.1. *The map g_α is convex for $\alpha > 1$, and concave for $\alpha < 1$.*

Proof. To prove this scholium, first note that $g'(z) = \frac{\alpha}{z^2 + z} (g(z) + g^2(z))$. This yields

$$g''(z) = -\alpha \frac{2z + 1}{(z^2 + z)^2} (g(z) + g^2(z)) + \frac{\alpha}{z^2 + z} (g'(z) + 2g'(z)g(z)).$$

If we replace in this last expression the value of $g'(z)$ in function of z and $g(z)$, we get

$$g''(z) = \alpha^2 \frac{g(z) + g^2(z)}{(z^2 + z)^2} \left(g(z) - \left(\frac{z}{\alpha} - \frac{\alpha - 1}{2} \right) \right).$$

Note that $\frac{z}{\alpha} - \frac{\alpha - 1}{2}$ is the asymptote of g close to $+\infty$. Then, the convexity of the map depends on the position of the graph with respect to the asymptote. It's convex when the graph is above the asymptote, and it's concave when the graph is below the asymptote. Now, recall that a convex map has a non-decreasing derivative, and a concave map has a non-increasing derivative. Therefore, easy considerations on the relative position of the graph with respect to the asymptote prove that the graph cannot cross the asymptote. Hence the map is convex for $\alpha > 1$, and concave for $\alpha < 1$. \square

Note that $g_\alpha(1) = \frac{1}{2^{\alpha-1}}$. Therefore, $g_\alpha(1) < 1$, for $\alpha > 1$, and $g_\beta(1) > 1$, for $\beta < 1$.

Lemma 4.2. *Let $(a_k)_{k \in \mathbb{N}}$ be a sequence of real numbers such that $a_0 = 0$, each a_{2k+1} is larger than 1, and all the even terms a_{2k} , $k > 0$, are positive and uniformly bounded away from zero. Then, the sequence of real numbers (r_k) defined by*

$$r_k = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_k}}}},$$

converges to a real number denoted by $[0, a_1, a_2, a_3, \dots]$, and we have

$$[0, a_1, a_2, a_3, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_k + \frac{1}{\ddots}}}}}.$$

Proof. Let $(a_k)_{k \in \mathbb{N}}$ be as in the assumptions. We define two new sequences $(p_k)_{k \in \mathbb{N}}$ and $(q_k)_{k \in \mathbb{N}}$, by induction:

$$p_0 = 0, \quad p_1 = 1, \quad q_0 = 1, \quad q_1 = a_1$$

$$\forall k \in \mathbb{N}, \quad p_{k+2} = a_{k+2}p_{k+1} + p_k, \quad q_{k+2} = a_{k+2}q_{k+1} + q_k.$$

It's easy to see, by induction, that for every $k > 0$, $q_k \geq 1$. Using $a_{2k+1} \geq 1$, we easily get $q_{2k+1} \geq k$, and then $q_{2k} \geq A.k$, where A is a positive lower bound for all the a_{2j} 's. Therefore, q_k goes to $+\infty$ as k increases to $+\infty$.

If we set $u_k = p_{k+1}q_k - p_kq_{k+1}$, then $u_{k+1} = -u_k$ for every k . We claim that $r_k = \frac{p_k}{q_k}$. Then, the two subsequences (r_{2k}) and (r_{2k+1}) are mutually adjacent and converge to the same limit. We leave to the reader to check that the even sequence (r_{2k}) increases and the odd sequence (r_{2k+1}) decreases. \square

We now consider real numbers, α in $[1, +\infty[$, β in $]0, 1]$, and the natural number $a \geq 0$. Given $\bar{x} = (\underbrace{0, \dots, 0}_{n_0}, \underbrace{1, \dots, 1}_{n_1}, \underbrace{0, \dots, 0}_{n_2}, 1, \dots) \in \Sigma = \{0, 1\}^{\mathbb{N}}$ we define a real number in $[0, 2^\beta - 1]$ in the following way:

$$\theta_{\alpha, \beta, a}(\bar{x}) = \frac{1}{\frac{(n_0 + 1)^\beta}{(n_0 + 2)^\beta - (n_0 + 1)^\beta} + \frac{1}{\frac{(n_1 + a)^\alpha}{(n_1 + a + 1)^\alpha - (n_1 + a)^\alpha} + \frac{1}{\frac{n_2^\beta}{(n_2 + 1)^\beta - n_2^\beta} + \frac{1}{\frac{(n_3 + a)^\alpha}{(n_3 + a + 1)^\alpha - (n_3 + a)^\alpha} + \dots}}}$$

We effectively claim (and let the reader check) that the sequence defined by $a_{2k} = g_\alpha(n_{2k-1} + a)$ and $a_{2k+1} = g_\beta(n_{2k})$ satisfies the properties of Lemma 4.2. Therefore the real number $[0, a_1, a_2, \dots]$ is well-defined.

We now claim that $\theta_{\alpha,\beta,a}(\bar{x})$ belongs to $[0, 2^\beta - 1]$. Indeed, the odd subsequence (r_{2k+1}) decreases and the even subsequence (r_{2k}) increases. To minimize the value of $\theta_{\alpha,\beta,a}(\bar{x})$, it is necessary and sufficient to maximize n_0 . On the other hand, to maximize the value of $\theta_{\alpha,\beta,a}(\bar{x})$, it's necessary and sufficient to minimize n_0 and to maximize n_1 . Therefore, for every \bar{x} ,

$$0 = \theta_{\alpha,\beta,a}(\bar{0}^\infty) \leq \theta_{\alpha,\beta,a}(\bar{x}) \leq \theta_{\alpha,\beta,a}(\bar{1}^\infty) = 2^\beta - 1.$$

Remark 2. The number a does not need to be in \mathbb{N} to define $\theta_{\alpha,\beta,a}$, but in \mathbb{R}^+ . This restriction will be explained later. Note also that θ is not an homeomorphism.

4.3. Identification of the Image Interval. In this subsection, we study a one-parameter family of Gauss-like transformations. The goal is to prove that for good parameters, the image $\theta_{\alpha,\beta,a}(\Sigma)$ is the interval $[0, 2^\beta - 1]$.

The case $\alpha > 1$. Convexity and the existence of the asymptote yield, for every $n \geq 1$,

$$(14) \quad 0 < g_\alpha(n+1) - g_\alpha(n) < \sup_{j>0} \{g_\alpha(j+1) - g_\alpha(j)\} = 1/\alpha.$$

For a positive y , we set $0 \leq r_\alpha(y) = y - g(n_y) \leq \frac{1}{\alpha}$.

If $\alpha = 1$, then $2^\alpha - 1 = 1$. In this case, $g(z) = z$ (for all z). Therefore, $g(n) = n$, and $r(x) = \frac{1}{x} - [\frac{1}{x}]$ is the usual fractional part of $1/x$.

For the case, $\alpha > 1$, we consider the new Gauss-like map $\phi_\alpha : (0, 2^\alpha - 1) \rightarrow [0, 1]$ given by

$$\phi_\alpha(x) = \frac{1}{x} - g_\alpha \left(\left[\frac{1}{(x+1)^{\frac{1}{\alpha}} - 1} \right] \right).$$

The case $\beta < 1$. We set

$$\phi_\beta(x) = \frac{1}{x} - g_\beta \left(\left[\frac{1}{(x+1)^{\frac{1}{\beta}} - 1} \right] \right).$$

Note that concavity and existence of the asymptote yield, for every $n \geq 1$,

$$(15) \quad \frac{1}{\beta} < g_\alpha(n+1) - g_\alpha(n) \leq \sup_{j>0} \{g_\alpha(j+1) - g_\alpha(j)\} = g_\beta(2) - g_\beta(1).$$

We now assume that α , β and a , satisfy

$$\frac{1}{(\frac{3}{2})^\beta - 1} - \frac{1}{2^\beta - 1} = \left(1 + \frac{1}{a+1}\right)^\alpha - 1. \quad (16a)$$

$$\frac{1}{\alpha} = 2^\beta - 1. \quad (16b)$$

This system of conditions is referred as (16). Note that this yields

$$(17) \quad \frac{1}{\frac{1}{(\frac{3}{2})^\beta - 1} + 0} = \frac{1}{\frac{1}{2^\beta - 1} + \frac{1}{\frac{1}{(1 + \frac{1}{a+1})^\alpha - 1} + 0}}.$$

Note also that if $\alpha = 1$, then $\beta = 1$. In this case $a = 0$.

We first check that conditions (16) are compatible with our assumptions $\alpha \geq 1$ and $\beta \leq 1$.

Note that $\beta \leq 1$ yields $2^\beta - 1 \leq 1$, and, then, we indeed have $\alpha \geq 1$.

One can ask: which values $a \in \mathbb{R}$ are possible?

Solving in a (the two equations) as a function of β , we have to consider the map

$$\beta \mapsto a(\beta) + 1 := \frac{1}{\left(\frac{1}{\frac{1}{(3/2)^\beta - 1} - \frac{1}{2^\beta - 1}} + 1 \right)^{2^\beta - 1} - 1}.$$

Lemma 4.3. *The map*

$$x \rightarrow \frac{1}{\left(\frac{1}{\frac{1}{(3/2)^x - 1} - \frac{1}{2^x - 1}} + 1 \right)^{2^x - 1} - 1}$$

is a decreasing bijection from $]0, 1]$ onto $[1, +\infty)$.

Proof. Once $(2^x - 1)$ is increasing, and $\frac{1}{\frac{1}{(3/2)^x - 1} - \frac{1}{2^x - 1}} + 1$ is bigger than 1; all we have to prove is that $\frac{1}{\frac{1}{(3/2)^x - 1} - \frac{1}{2^x - 1}} + 1$ is also increasing.

Given $f : [0, 1] \rightarrow \mathbb{R}$ and $g : [0, 1] \rightarrow \mathbb{R}$, one can ask when $\frac{1}{f(x)} - \frac{1}{g(x)}$ is decreasing? Taking derivative we get the condition

$$\frac{f'(x)}{f(x)^2} > \frac{g'(x)}{g(x)^2}$$

Suppose $f(x) = ((3/2)^x - 1)$, then the first term is

$$\frac{\log(3/2) (3/2)^x}{((3/2)^x - 1)^2}.$$

Suppose $g(x) = (2^x - 1)$, then the second term is

$$\frac{\log 2 \cdot 2^x}{(2^x - 1)^2}.$$

We claim that, for all $x \in [0, 1]$,

$$\frac{\log(3/2)}{((3/2)^x - 1)(1 - (3/2)^{-x})} = \frac{\log(3/2) (3/2)^x}{((3/2)^x - 1)^2} > \frac{\log 2 \cdot 2^x}{(2^x - 1)^2} = \frac{\log 2}{(2^x - 1)(1 - 2^{-x})}.$$

The above means, for all $x \in [0, 1]$,

$$v(x) = \log(3/2) (2^x - 2 + 2^{-x}) > \log 2 ((3/2)^x - 2 + (3/2)^{-x}) = u(x).$$

Note that $v(0) = 0 = u(0)$.

As

$$v'(x) = \log(3/2) \log 2 \cdot 2^x - \log(3/2) \log 2 \cdot 2^{-x},$$

and

$$u'(x) = \log(3/2) \log 2 (3/2)^x - \log(3/2) \log 2 (3/2)^{-x}.$$

For any non-negative x , we have $v'(x) > u'(x)$ and the claim follows. Therefore,

$$x \mapsto \frac{1}{((3/2)^x - 1)} - \frac{1}{(2^x - 1)}$$

is decreasing on $[0, 1]$. This means that

$$x \mapsto \frac{1}{\frac{1}{((3/2)^x - 1)} - \frac{1}{(2^x - 1)} + 1}$$

is increasing.

Moreover, close to 1 we have $a(x) = (x-1)(2 \ln 2 - 4 \ln^2(2) - 6 \ln \frac{3}{2}) + O((x-1)^2)$.

Close to 0 we have $a(x) = \frac{\frac{1}{\ln \frac{3}{2}} - \frac{1}{\ln 2}}{x^2 \ln 2} + C \cdot \frac{1}{x} + O(1)$.

□

From the lemma above we get the property that each positive integer value of a can be reached. In this way, several renormalization operators, with different values $a \in \mathbb{N}$, can be considered in our future reasoning. For each such value a , we have the corresponding values α_a and β_a . We point out, however, that it also has meaning to consider real values of a (any positive real is possible) in several of our results (which are not related to the renormalization operator for the shift)

In the following, we however prefer to keep β as parameter, because it gives the length of the interval where the dynamic works :

Proposition 4.4. *With our new conditions on α , β and a , the map $\theta_{\alpha, \beta, a}$, from Σ_2^+ into the interval $[0, 2^\beta - 1]$ is onto.*

Proof. For a given x in $[0, 2^\beta - 1]$, we want to exhibit a sequence n_0, n_1, n_2, \dots of integers, possibly equal to $+\infty$ (in that case the sequence stops), all positive, except n_0 which is non-negative, such that

$$x = \frac{1}{\frac{(n_0 + 1)^\beta}{(n_0 + 2)^\beta - (n_0 + 1)^\beta} + \frac{1}{\frac{(n_1 + a)^\alpha}{(n_1 + a + 1)^\alpha - (n_1 + a)^\alpha} + \frac{1}{\frac{n_2^\beta}{(n_2 + 1)^\beta - n_2^\beta} + \frac{1}{\frac{(n_3 + a)^\alpha}{(n_3 + a + 1)^\alpha - (n_3 + a)^\alpha} + \frac{1}{\dots}}}},$$

If such a sequence exists, then $\frac{1}{x} = g_\beta(n_0 + 1) + r_0(x)$. We may, for instance choose $n_0 + 1$ as the integer $n_{\frac{1}{x}}$ (see (13) but for g_β). In that way, (16a) yields that $r_0(x) \leq (1 + \frac{1}{a+1})^\alpha - 1$. Moreover, $x \leq 2^\beta - 1$ yields $\frac{1}{x} \geq g_\beta(1)$, hence $n_0 \geq 0$. We thus have now to iterate this construction by induction. Clearly $\frac{1}{r_0(x)} \geq g_\alpha(1 + a)$, and we can find some $n_1 \geq 1$ such that

$$\frac{1}{r_0(x)} = g_\alpha(n_1 + a) + r_1(x).$$

We then have $r_1(x) \leq \frac{1}{\alpha} = 2^\beta - 1$, and we can iterate this process.

□

We now explain what are the points with finite fractional expansion. In the construction the process stops if and only if some rest $r_i(x)$ equals 0. If i is even, we are dealing with the maps g_β and ϕ_β . The symbolic representation of x is $\bar{x} = (\underbrace{0, \dots, 0}_{n_0}, \underbrace{1, \dots, 1}_{n_1}, \dots, \underbrace{0, \dots, 0}_{n_i}, \underbrace{1, \dots, 1}_{n_{i+1}}, \dots)$, and ultimately equals 1. Just at the right side of x , in $x + 0$, points have one zero less in their symbolic representation in Σ for the i^{th} -block, then one 1 and arbitrarily long string of 0's. Just at the left side of x , in $x - 0$, points have n_i 0's and arbitrarily long string of 1.

If i is odd, we are dealing with maps g_{α} and ϕ_β . Things are similar, except we have to exchange left side with right side, and 0 with 1.

4.4. An associated global transformation on $[0, 2^\beta - 1]$. Let $\alpha \geq 1$, $\beta \leq 1$ and a such that (16a) and (16b) are satisfied. Given $\bar{x} = (\underbrace{0, \dots, 0}_{n_0}, \underbrace{1, \dots, 1}_{n_1}, \underbrace{0, \dots, 0}_{n_2}, \underbrace{1, \dots, 1}_{n_3}, \dots) \in \Sigma$ we set

$$\hat{\theta}_\beta(\bar{x}) := \theta_{\alpha, \beta, a}(\bar{x}) = \frac{1}{g_\beta(n_0 + 1) + \frac{1}{g_\alpha(n_1 + a) + \frac{1}{g_\beta(n_2) + \frac{1}{g_\alpha(n_3 + a) + \dots}}}}$$

In the above expression, we can assume the possibility $n_0 = 0$. If one consider in $\{0, 1\}^\mathbb{N}$ the lexicographic order, then the global transformation $\hat{\theta}_\beta$ is non-decreasing in each cylinder $\bar{0}$ and $\bar{1}$.

Note that $\hat{\theta}_\beta(\bar{0}^\infty) = 0$, and $\hat{\theta}_\beta(\bar{1}^\infty) = 2^\beta - 1$. Moreover, (17) yields

$$\hat{\theta}_\beta(01^\infty) = \frac{1}{\frac{1}{(\frac{3}{2})^\beta - 1} + 0} = \frac{1}{\frac{1}{2^\beta - 1} + \frac{1}{\frac{1}{(1 + \frac{1}{a+1})^\alpha - 1} + 0}} = \hat{\theta}_\beta(010^\infty)$$

We then define the transformation $f_\beta : [0, 2^\beta - 1] \rightarrow [0, 2^\beta - 1]$ by

$$f_\beta(x) = [\hat{\theta}_\beta \circ \sigma \circ \hat{\theta}_\beta^{-1}](x).$$

Namely, if

$$x = \frac{1}{\frac{(n_0+1)^\beta}{(n_0+2)^\beta - (n_0+1)^\beta} + \frac{1}{\frac{(n_1+a)^\alpha}{(n_1+a+1)^\alpha - (n_1+a)^\alpha} + \frac{1}{\frac{n_2^\beta}{(n_2+1)^\beta - n_2^\beta} + \frac{1}{\frac{(n_3+a)^\alpha}{(n_3+a+1)^\alpha - (n_3+a)^\alpha} + \dots}}}},$$

and if $n_0 > 0$, then

$$f_\beta(x) = \frac{1}{\frac{n_0^\beta}{(n_0+1)^\beta - n_0^\beta} + \frac{1}{\frac{(n_1+a)^\alpha}{(n_1+a+1)^\alpha - (n_1+a)^\alpha} + \frac{1}{\frac{n_2^\beta}{(n_2+1)^\beta - n_2^\beta} + \frac{1}{\frac{(n_3+a)^\alpha}{(n_3+a+1)^\alpha - (n_3+a)^\alpha} + \dots}}}$$

If $n_0 = 0$, and if $n_1 > 1$, then

$$f_\beta(x) = \frac{1}{\frac{1}{2^\beta - 1} + \frac{1}{\frac{(n_1+a-1)^\alpha}{(n_1+a)^\alpha - (n_1+a-1)^\alpha} + \frac{1}{\frac{n_2^\beta}{(n_2+1)^\beta - n_2^\beta} + \frac{1}{\frac{(n_3+a)^\alpha}{(n_3+a+1)^\alpha - (n_3+a)^\alpha} + \dots}}}$$

If $n_0 = 0$ and $n_1 = 1$, then

$$f_\beta(x) = \frac{1}{\frac{n_2^\beta}{(n_2+1)^\beta - n_2^\beta} + \frac{1}{\frac{(n_3+a)^\alpha}{(n_3+a+1)^\alpha - (n_3+a)^\alpha} + \frac{1}{\frac{n_4^\beta}{(n_4+1)^\beta - n_4^\beta} + \frac{1}{\frac{(n_5+a)^\alpha}{(n_5+a+1)^\alpha - (n_5+a)^\alpha} + \dots}}}$$

When $\beta = 1$, the transformation f_β plays a similar role of F in [2] page 3540. Note that $f_\beta(0) = 0$, and $f_\beta(1) = 1$.

Proposition 4.5. *The map f_β is increasing and differentiable in each interval $\left[0, \left(\frac{3}{2}\right)^\beta - 1\right)$ and $\left(\left(\frac{3}{2}\right)^\beta - 1, 1\right]$.*

Proof. First consider $x \in \left[0, \left(\frac{3}{2}\right)^\beta - 1\right)$. If x does not have a finite continuous expansion, then $f'_\beta(x) = \frac{f_\beta(x)^2}{x^2}$.

Indeed, suppose $x = \hat{\theta}_\beta(\bar{x}) = (\underbrace{0, \dots, 0}_{n_0}, \underbrace{1, \dots, 1}_{n_1}, \underbrace{0, \dots, 0}_{n_2}, 1, \dots) \in \Sigma$, and take

$y = \hat{\theta}_\beta(\bar{y})$, with y close to x in $\left[0, \left(\frac{3}{2}\right)^\beta - 1\right)$. Note that $n_0 > 0$.

As a is an integer, discontinuities of the fractional expansion only appear for a fixed countable set of points (whatever the “level” they appear). Assume x is not such a point. Then, \bar{x} and \bar{y} are close in Σ . Using the above expansion suppose $x = \frac{1}{u+b}$ and $y = \frac{1}{u+b'}$, where $u = g_\beta(n_0 + 1)$.

Therefore, $f_\beta(x) = \frac{1}{\zeta+b}$ and $f_\beta(y) = \frac{1}{\zeta+b'}$. Finally,

$$\frac{f_\beta(x) - f_\beta(y)}{x - y} = \frac{\frac{1}{\zeta+b} - \frac{1}{\zeta+b'}}{\frac{1}{u+b} - \frac{1}{u+b'}} = \frac{(u+b)(u+b')}{(\zeta+b)(\zeta+b')}.$$

When $y \rightarrow x$ we get $b' \rightarrow b$. Then, we get the expression

$$f'_\beta(x) = \frac{f_\beta(x)^2}{x^2}.$$

We claim that this also holds if x is a discontinuity point, but if y goes to $x + 0$ or $x - 0$ (depending if the tail of \bar{x} is only 0 or only 1). In that case we just have a left or right derivative.

Let us now study the other side. Note that we only consider the case $n_0 > 1$ because the map is not continuous in $\left(\frac{3}{2}\right)^\beta - 1$. We have

$$\frac{f_\beta(x) - f_\beta(y)}{x - y} = \frac{\frac{1}{\zeta+b} - \frac{1}{\zeta'+b'}}{\frac{1}{u+b} - \frac{1}{u'+b'}} = \frac{(u+b)(u'+b')}{(\zeta+b)(\zeta'+b')},$$

where $u' \neq u$ and $\zeta' \neq \zeta$ if and only if $n_1 = +\infty$. When $y \rightarrow x$, we do not claim that $u' \rightarrow u$ and $b' \rightarrow b$, but the fact that $\hat{\theta}_\beta$ is onto yields that in \mathbb{R} , $u' + b' \rightarrow u + b$ and $\zeta' + b' \rightarrow \zeta + b$.

Therefore, for any x in $\left[0, \left(\frac{3}{2}\right)^\beta - 1\right)$,

$$f'_\beta(x) = \frac{f_\beta(x)^2}{x^2}.$$

The case x in $\left[\left(\frac{3}{2}\right)^\beta - 1, 1\right]$ is similar. Note that this interval is also

$$\left[\frac{1}{\frac{1}{2^\beta - 1} + \left(1 + \frac{1}{a+1}\right)^\alpha - 1}, 1\right],$$

and we have to exchange the variable x with $1 - x$. □

An easy calculation shows that on $\left[0, \left(\frac{3}{2}\right)^\beta - 1\right)$, $f_\beta(x)$ is on the form $f_\beta(x) = \frac{x}{1+cx}$. To find c we use the boundary condition $f_\beta\left(\left(\frac{3}{2}\right)^\beta - 1\right) = 2^\beta - 1$. Then (16a) yields

$$-c = \frac{1}{\left(\frac{3}{2}\right)^\beta - 1} - \frac{1}{2^\beta - 1} = \left(1 + \frac{1}{a+1}\right)^\alpha - 1.$$

When $a = 0$ we get $c = -1$.

In a similar way, we can find a value d such that $f_\beta(x) = \frac{d+(1-d)x}{1+d-dx}$, for $x \in \left[\left(\frac{3}{2}\right)^\beta - 1, 1\right]$.

In this way we obtain f_β as a new kind of Manneville-Pomeau-like map. The values of c and d depend on a , α and β . Note that the jet in each of the two fixed points of this map is associated to a different parameter, namely, depend respectively on c and d .

Remark 3. As we said in the introduction, the condition $a \in \mathbb{N}$ does not seem to be necessary. We can define $\theta_{\alpha,\beta,a}$ with $a \in \mathbb{R}_+$, which give us continuous $\alpha(a)$ and $\beta(a)$. We however recall that $a \in \mathbb{N}$ was needed to define the renormalization in the shift.

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